CHAPTER 3
QUEUEING ANALYSIS FOR THE $M^{[x]}/D/1$ QUEUE

In this chapter, we present the analysis of the $M^{[x]}/D/1/\infty$ queue, i.e. the $M/D/1$ queue with group arrivals and unlimited buffer. The analysis forms the basis for investigating the variants of the queue. The variants, namely the $M^{[x]}/D/1$ queue with threshold and the $M^{[x]}/D/1/B$ queue, are used to model the performance of an ATM switch operating with the PDS and with the Cell Discard Strategy respectively. Here, we outline the necessary techniques for the modelling and analysis. An algorithm is also established to calculate the equilibrium queue occupancy distribution at an arbitrary instant for the $M^{[x]}/D/1/\infty$ queue.

The approach to calculate the distribution for the $M/D/1$ queue with no group arrivals would be to study the imbedded Markov chain to yield the equilibrium distribution for the number of customers left behind by a departing customer. This distribution in turn would be equal to that of at an arbitrary time [10](pp 176). However, in the case of group arrivals, the distribution at a departure instant is not equal to that of at an arbitrary instant. Fortunately though, they are related through the distribution of backward recurrence time to the most recent departure. The relationship may be obtained by comparing the results of the method of imbedded Markov chain and the method of supplementary variables. The latter method provides the equilibrium distribution for the number of customers at an arbitrary instant.

In Section 3.1, we describe the $M/D/1$ queue with group arrivals. In Section 3.2, we use the method of imbedded Markov chain to obtain the distribution of the number of cells left behind by a departing cell. In Section 3.3, the method of supplementary variables is used to obtain the equilibrium probability distribution for the number of cells at an arbitrary time. Also explored in this section is the relationship between the equilibrium probability distribution for the number of cells left behind by a departing cell and that for the number of cells at an arbitrary time. In Section 3.4, an algorithm is given to calculate the required distributions. Finally in Section 3.5, we summarize the chapter.

3.1 Queue Description

The $M^{[x]}/D/1$ queue is a variant of the $M/D/1$ queue in that in the former a group of customers is allowed to arrive at each arrival instant whereas in the latter only one customer may arrive at a given instant. Group arrivals occur according to a Poisson process. The number of customers contained in each group is independent of the arrival process. Furthermore, group sizes are independent and identically distributed according to a specified distribution. Customers in a group are then served one by one. Such a queue is considered in [10](pp 235), [11](pp 44-51) and [13](pp 385-387). A group of customers may also be referred to as a batch of customers. In this document, these terms are used interchangeably.

In this section, we first describe the arrival process involving group arrivals. Next, we consider two basic event sequences that typically characterize the behaviour of the queue. Then, we explore the relationship between the arrival process and the service process.

In exploring the relationship between the arrival process and the service process, we first look at the distribution for the number of arrivals during a service period. We then consider the distribution for the number of customers left behind at the end of the first service period which initiates a busy period.

3.1.1 Group Arrival Process

Let λ be the average arrival rate of groups and $\bar{x}$ be the time required for serving one customer. Let G be the random variable representing the size of a group and g_i be the probability that the group size G is i, where $i = 1,2, \ldots$. Let $G(z)$ and g be the probability generating function *(PGF)* and the mean of the group size, respectively. Then, we have

$$g_i = P[G = i], \tag{1}$$

$$G(z) = \sum_{k=1}^{\infty} g_k z^k, \tag{2}$$

$$g = E[G] = G'(1). \tag{3}$$

Recall that in a single server queue, the server utilization, ρ, is the product of the average arrival rate of customers and the average service time. In this particular case, the average arrival rate of customers is the product of the average arrival rate of groups and the average group size. Thus, the server utilization for this queue is defined as

$$\rho = \lambda \bar{x} g. \tag{4}$$

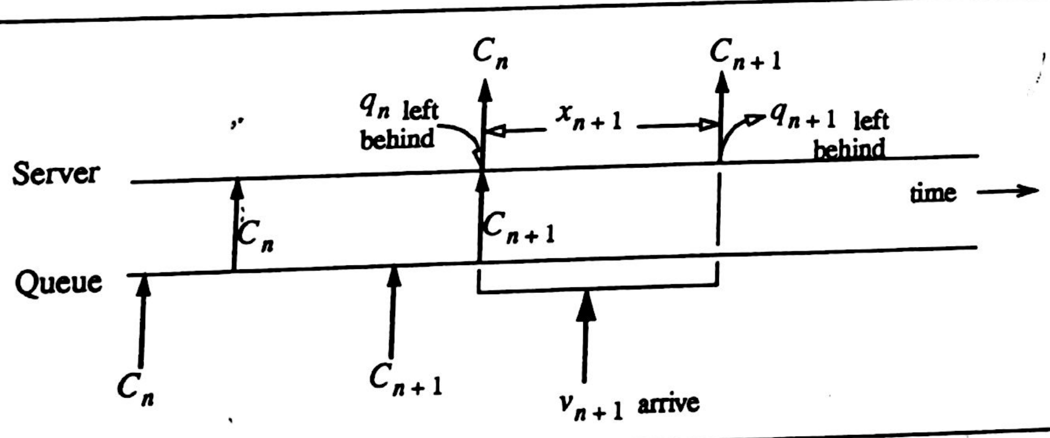
3.1.2 Basic Event Sequences

Next, we consider two basic event sequences that typically characterize the behaviour of the queue. In the subsequent discussion, let C_n denote the n^{th} customer to enter the queue and define the following random variables: $\bar{x}_n$ to represent the service time for C_n; t_n to indicate the departure instant of C_n; q_n to denote the number of customers left behind by the departure of C_n from service; and finally v_n to represent the number of customers arriving during the service of C_n.

The first event sequence refers to the case where the service of a customer, C_{n+1}, immediately follows that of another, C_n, as shown in Figure 3.1. This situation necessarily requires the number of customers left behind by the departure of C_n is always greater than zero, that is $q_n > 0$. Consequently, the number of customers left behind by the departure of C_{n+1} may be calculated as that of C_n less 1 (departing customer) plus the number of customers arrive during the service interval $\bar{x}_{n+1}$ and can be expressed as

$$q_n > 0 \quad \Rightarrow \quad q_{n+1} = q_n - 1 + v_{n+1}. \tag{5}$$

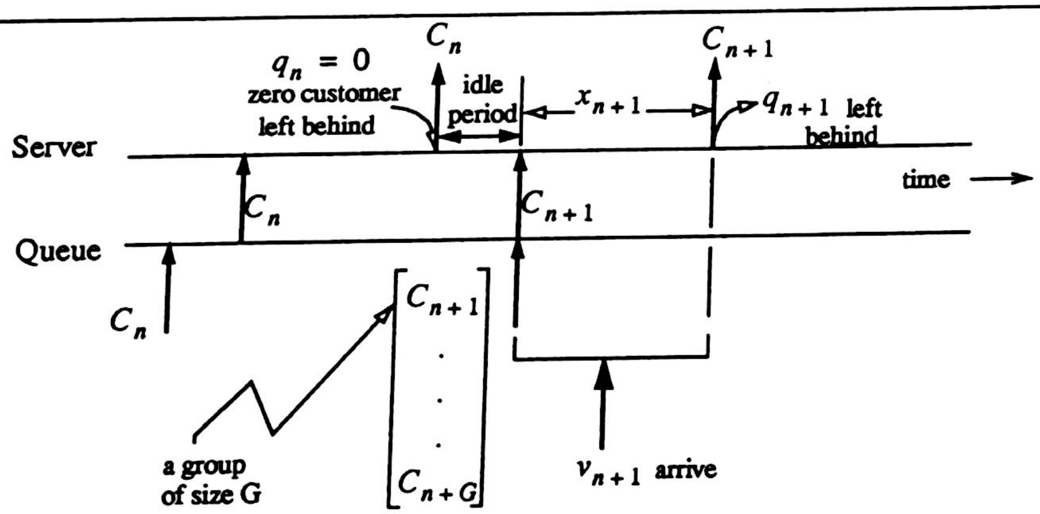
Figure 3.1: Sequence of events within a service period within a busy period



The second event sequence refers to the case where the service of a customer, C_{n+1}, is separated from that of another, C_n, by an idle period, as illustrated in Figure 3.2. This situation results whenever the number of customers left behind by the departure of C_n is equal to zero, that is $q_n = 0$. Noting that then C_{n+1} is the first customer of the group of size G to be served, the number of customers left behind by the departure of C_{n+1} is equal to the number of customers arrive during its service plus the rest of the group each of those customers has arrived with and can be expressed as

$$q_n = 0 \quad \Rightarrow \quad q_{n+1} = v_{n+1} + G - 1. \tag{6}$$

Figure 3.2: Sequence of events during the initiation of a busy period



We recall that the arrival process is independent of the number of customers in the queue. Similarly, $\bar{x}_n$ is independent of n and is constant. Therefore, v_n, the number of arrivals during the service time $\bar{x}_n$ depends only upon the duration $\bar{x}$ and not upon n at all. We may therefore dispense with the subscripts on v_n and $\bar{x}_n$, replacing them with the random variables v and $\bar{x}$.

3.1.3 Arrivals during a service period

Let N represent the number of group arrivals in a service period of length $\bar{x}$. With Poisson arrivals, we have

$$P[N = k] = \frac{(\lambda\bar{x})^k}{k!}e^{-\lambda\bar{x}} \qquad \text{for } k = 0,1,2,... \tag{7}$$

where λ is the average group arrival rate. Consequently, the number of customers arrives in a service period of length $\bar{x}$ is given by

$$v = G_1 + G_2 + \ldots + G_N \tag{8}$$

where G_j represents the number of customers in the j^{th} group that arrives in the service period. That is, the number of customer arrivals in one service period is equal to a random sum of the random variables each representing a group size distribution.

Let α_k denote the probability that the number of customers which arrives in a service period of length x is equal to k. That is,

$$\alpha_k = P[v = k]. \tag{9}$$

Note that, the event of no customer arrivals is equivalent to the event of no group arrivals. Therefore, we have

$$\alpha_0 = P[N = 0] = e^{-\lambda\bar{x}} \tag{10}$$

where we make use of (7). For $k \geq 1$, that is when there is at least one arrival, we can express α_k as

$$\alpha_k = \sum_{n=1}^{k} P[v = k|N = n]P[N = n] \tag{11}$$

where we have applied the total probability theorem by conditioning v upon the number of groups which have arrived. Note that, $v = k$ may consist of k groups, each of size one. Similarly, it may consist of only one group of size k. Accordingly, by letting

$$g_k^{(n)} = \underbrace{g_k \otimes g_k \otimes \ldots \otimes g_k}_{n \text{ times}} \tag{12}$$

denote the n-fold convolution of g_k, we have

$$P[v = k | N = n] = P[G_1 + G_2 + G_3 + \ldots + G_n = k] = g_k^{(n)} \tag{13}$$

as given in [10](pp 375-377) for any sum of independent and identically distributed random variables. Furthermore, $g_k^{(n)}$ can be recursively calculated as

$$\begin{aligned} g_k^{(1)} &= g_k \\ g_k^{(n)} &= \sum_{i=1}^{k-1} g_i g_{k-i}^{(n-1)} \qquad n > 1 \end{aligned} \tag{14}$$

which is also given in [14](pp 133). Next, substituting (13) and (7) into (11), we get

$$\alpha_k = \sum_{n=1}^{k} \frac{(\lambda\bar{x})^n}{n!} e^{-\lambda\bar{x}} g_k^{(n)} \qquad k \geq 1. \tag{15}$$

Finally, the probability generating function of v, $V(z)$, is derived in Appendix A for a general service time distribution. For the case of constant service period $\bar{x}$, it reduces to

$$V(z) \equiv \sum_{k=0}^{\infty} P[v = k] z^k = e^{-\lambda\bar{x}[1 - G(z)]}. \tag{16}$$

3.1.4 Arrivals at the start of a busy period

As illustrated in Figure 3.2, the number of customers left behind by the first customer served at the start of a busy period, not only contains the customers that arrived during its service period but also contains the rest of the customers within its group.

Let β_k be the probability that the number of customers left behind by the departure of the first customer, whose group initiated a busy period, is equal to k. Namely,

$$\beta_k = P[v + G - 1 = k] \tag{17}$$

Then β_k can be expressed as

$$\begin{aligned}\beta_k &= \sum_{m=1}^{k+1} P[v = k - m + 1 | G = m] P[G = m] \\ &= \sum_{m=1}^{k+1} g_m \alpha_{k-m+1}\end{aligned} \tag{18}$$

where we conditioned v upon the size of the group which initiates a busy period. Note that, the minimum size of the group that initiates a busy period must be one for all values of k and it cannot exceed $k+1$ for $v + G - 1 = k$.

3.2 Queue Occupancy Distribution at Departure Instants

The traditional way of determining the equilibrium distribution for the number of customers left behind by a departing customer is to establish a relationship between successive departure instants and exploit that relationship to obtain the probability generating function of the required distribution.

Although the probability generating function of the distribution provides the necessary information in a closed form, it may not be suitable for efficient calculation of numerical probabilities. Instead, in numerically calculating the required distribution, the state transition probabilities may equivalently be used to make use of the special structure common to all the $M/G/1$ type stochastic matrices. The latter approach requires the probability of having an empty queue at departure instants which is readily available from the former method.

Hence, in this section, we first derive the probability generating function of the equilibrium queue occupancy distribution at departure instants. Next, we determine the probability of having an empty queue at customer departures. Then we examine the structure of the transition probability matrix and specify an algorithm for calculating stationary probabilities in a recursive manner.

3.2.1 Probability generating function of the distribution

In Section 3.1, we have established the fundamental equations (5) and (6) that typically characterize the behaviour of the queue. Here, we combine those equations as

$$q_{n+1} = q_n - \Delta q_n + v_{n+1} + (1 - \Delta q_n)(G - 1) \tag{19}$$

where

$$\begin{aligned} \Delta k &= 1 \qquad k > 0 \\ &= 0 \qquad k \leq 0 \end{aligned} \tag{20}$$

Next, we define the probability generating function for the random variable q_n as

$$Q_n^d(z) = E[z^{q_n}] = \sum_{k=0}^{\infty} P[q_n = k]z^k \quad (21)$$

and that for the limiting random variable $q = \lim_{n \to \infty} q_n$ as

$$Q^d(z) = \lim_{n \to \infty} Q_n^d(z) = E[z^q] = \sum_{k=0}^{\infty} d_k z^k \quad (22)$$

where d_k denotes the equilibrium probability that a departure leaves k customers behind, namely

$$d_k = P[q = k]. \quad (23)$$

By applying the definition given in (21) to the random variable q_{n+1}, we have

$$Q_{n+1}^d(z) = E[z^{q_n - \Delta q_n + v_{n+1} + (1 - \Delta q_n)(G - 1)}] \quad (24)$$

where we make use of (19). Note that the number of customers that arrive during the service period of the $(n+1)^{st}$ customer, v_{n+1}, is independent of the number of customers in the system at the previous departure, q_n. Therefore, we can rewrite (24) as

$$\begin{aligned} Q_{n+1}^d(z) &= E[z^{v_{n+1}}]E[z^{q_n - \Delta q_n + (1 - \Delta q_n)(G - 1)}] \\ &= V(z)E[z^{q_n - \Delta q_n + (1 - \Delta q_n)(G - 1)}] \end{aligned} \quad (25)$$

where we also make use of the fact that the distribution for the number of arrivals in a service period is independent of n. Expanding the second term on the right hand side in (25), we have

$$E[z^{q_n - \Delta q_n + (1 - \Delta q_n)(G - 1)}] = \sum_{k=0}^{\infty} \sum_{i=1}^{\infty} P[q_n = k, G = i] z^{k - \Delta k + (1 - \Delta k)(i - 1)}. \tag{26}$$

Noting the definition of Δk given in (20) leads to

$$\begin{aligned} &E[z^{q_n - \Delta q_n + (1 - \Delta q_n)(G - 1)}] \\ &= P[q_n = 0] \sum_{i=1}^{\infty} P[G = i] z^{(i-1)} + \sum_{k=1}^{\infty} P[q_n = k] z^{k-1} \sum_{i=1}^{\infty} P[G = i] \\ &= \frac{1}{z} P[q_n = 0] G(z) + \frac{1}{z} [Q_n^d(z) - P[q_n = 0]] \end{aligned} \tag{27}$$

where we make use of the fact that the group size is independent of the number of customers left behind upon a departure. Substituting (27) back into (25) leads to

$$Q_{n+1}^d(z) = \frac{1}{z} P[q_n = 0] G(z) V(z) + \frac{1}{z} Q_n^d(z) V(z) - \frac{1}{z} P[q_n = 0] V(z). \tag{28}$$

We can now set the limit as $n \to \infty$ and solve for $Q^d(z)$ to obtain

$$Q^d(z) = \frac{d_0 V(z)[1 - G(z)]}{V(z) - z} \tag{29}$$

where we make use of (22) and (23). The probability that a departing customer leaves behind an empty system, d_0, can be determined from the normalization condition, namely $Q^d(z)|_{z=1} = 1$. As carried out in Appendix B, we have

$$d_0 = \frac{1 - \lambda \bar{x} g}{g}. \tag{30}$$

Finally, substituting (30) into (29), we have the probability generating function of the equilibrium queue occupancy distribution at departure instants as

$$Q^d(z) = \frac{(1-\rho)}{g} \frac{[1 - G(z)]e^{-\lambda \bar{x}[1 - G(z)]}}{e^{-\lambda \bar{x}[1 - G(z)]} - z} \tag{31}$$

where we make use of (4) and (16).

3.2.2 Transition probability matrix and numerical calculations

Next, we find the transition probabilities that describe the behaviour of the Markov chain formed at departure instants. The one-step transition probabilities are defined as

$$p_{ij} \equiv P[q_{n+1} = j | q_n = i]. \tag{32}$$

p_{ij} can be interpreted as the probability of C_{n+1} leaving j customers behind given that C_n has left i customers behind. From the two fundamental equations (5) and (6) that typically characterize the behaviour of the queue, we first observe that $q_{n+1} < q_n - 1$ is an impossible situation and leads to $p_{ij} = 0$ for $j < i - 1$. Next, note that p_{0j} represents a transition from an empty queue to a state where j customers are left behind at the first departure instant of a busy period. This case has been described by the relation given in (6) and hence $p_{0j} = \beta_j$ for $j \geq 0$. The remaining transitions are only due to the arrivals v_{n+1}. For $q_n > 0$ and $q_{n+1} \geq q_n - 1$, we require $j - i + 1$ customers to arrive during the corresponding service period to make up the j customers at the departure instant from a previous value of i. Therefore, $p_{ij} = \alpha_{j-i+1}$ for $j \geq i-1$. Hence the matrix of transition probabilities $P = [p_{ij}]$ takes the following form:

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 & \dots \\ \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \dots \\ 0 & \alpha_0 & \alpha_1 & \alpha_2 & \dots \\ 0 & 0 & \alpha_0 & \alpha_1 & \dots \\ 0 & 0 & 0 & \alpha_0 & \dots \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{bmatrix} \end{array} \tag{33}$$

As suggested in [10](pp 179), equilibrium probabilities may be calculated using the vector equation

$$\vec{d} = \vec{d}P \tag{34}$$

where $\vec{d} = [d_0, d_1, d_2, \dots, d_k, \dots]$ whose k^{th} component denotes the equilibrium probability of having k customers in the system at departure instants. Noting that the transition probability matrix is an upper diagonal matrix, the equation (34) can be re-written in the expanded form as

$$d_k = \beta_k d_o + \sum_{i=1}^{k+1} \alpha_{k-i+1} d_i \qquad k \geq 0 \tag{35}$$

or equivalently solving for d_{k+1}

$$d_{k+1} = \frac{1}{\alpha_0}\left(d_k - \beta_k d_o - \sum_{i=1}^{k} \alpha_{k+1-i} d_i\right) \qquad k \geq 0. \tag{36}$$

Therefore, the equilibrium probabilities, starting from d_1, can be recursively calculated provided that d_0 is known. Fortunately though, the calculations may be carried out without the prior knowledge of d_0. As shown in Appendix C, the equilibrium probability that a departure leaves k customers behind is of the form

$$d_k = d_0 f_k \qquad k \geq 0 \tag{37}$$

where f_k satisfies the recursive relationship

$$f_0 = 1$$

$$f_{k+1} = \frac{1}{\alpha_0}\left(f_k - \beta_k - \sum_{i=1}^{k} f_i \alpha_{k+1-i}\right) \qquad k \geq 0 \tag{38}$$

which can be used to calculate f_k up to any values of k without the knowledge of d_0. Once d_0 is known, d_k can be calculated according to (37).

The recursion formula given in (38) may suffer from loss of significance in computer implementations of practical interest since the quantity inside the brackets is the difference of the positive quantity f_k, which becomes small for large k, and a positive sum of comparable magnitude. However, it is possible to transform the relation into a mathematically equivalent form which is highly stable and ideally suited for numerical computation [15](pp 16). The details of the transformation have been provided in Appendix D. To that end, we introduce two new quantities $\hat{\alpha}_k$ and $\hat{\beta}_k$

$$\hat{\alpha}_k = 1 - \sum_{i=0}^{k} \alpha_i, \qquad \hat{\beta}_k = 1 - \sum_{i=0}^{k} \beta_i, \qquad k \geq 0 \tag{39}$$

which are respectively defined in terms of α_k and β_k. As shown in Appendix D, f_k may then be accurately computed as

$$\begin{aligned} f_0 &= 1 \\ f_1 &= \frac{1}{\alpha_0}\hat{\beta}_0 \\ f_k &= \frac{1}{\alpha_0}\left(\hat{\beta}_{k-1} + \sum_{i=1}^{k-1} \hat{\alpha}_{k-i} f_i\right) \qquad k \geq 2. \end{aligned} \tag{40}$$

3.3 Queue Occupancy at Arbitrary Instants

As mentioned earlier, in the case of group arrivals, the distribution for the number of customers left behind by a departing customer is no longer equal to that of at an arbitrary instant. In determining the distribution for the number of customers present in the system at an arbitrary instant, t, one needs to consider not only the number of customers present in the system at time t but also the elapsed time since the start of the most recent service. The introduction of the latter variable is known as the method of supplementary variables.

In this section, we extend the derivation given in [11](pp 58-70) to the case of group arrivals. First, we define the stationary joint distribution for the number of customers present in the queue and the elapsed service time. Next, we derive the relations that the stationary distribution satisfies. Finally, the probability generating function for the distribution of the number of customers present in the queue at an arbitrary time is obtained by applying transform techniques to the derived relations.

3.3.1 Queue size and elapsed service time

Let L_t be the number of customers in the system at time t. Further, let X_t^- denote the amount of service already received by the customer in service at time t. Although L_t is not a Markov process in general, the process $[L_t, X_t^-]$ is, as the current value of the latter process summarizes the entire past history of its motion as far as its future behaviour is concerned. We assume that $X_t^- = 0$ when $L_t = 0$. Let us define the stationary joint distribution for the number of customers present in the queue and the elapsed service time as

$$p_0 \equiv \lim_{t \to \infty} Pr\{L_t = 0\}, \tag{41}$$

$$p_k(x)dx \equiv \lim_{t \to \infty} Pr\{L_t = k, X_t^- \in x_{dx}\} \qquad k \geq 1, x \geq 0 \tag{42}$$

where $\{Y \in y_{dy}\}$ denotes $\{y < Y \leq y + dy\}$. Also, we introduce the *hazard rate* or the *age-specific failure rate* $\bar{b}(x)$ as

$$\bar{b}(x) \equiv \frac{b(x)}{1 - B(x)} \tag{43}$$

which is the density function for the service time X on the condition that $X > x$, that is,

$$\bar{b}(x)dx = Pr\{X \in x_{dx} | X > x\}. \tag{44}$$

We first consider $p_k(x + \Delta x)$ for $k \geq 1$. The event $\{L_{t+\Delta x} = k, X_{t+\Delta x}^- \in (x + \Delta x)_{dx}\}$ occurs either when $\{L_t = k, X_t^- \in x_{dx}\}$ and there are no arrivals during Δx, or when $\{L_t = k - i, X_t^- \in x_{dx}\}$ and there is an arrival of size i during Δx. A further requirement in either case is that there are no service completions during Δx. Thus, we have

$$p_k(x + \Delta x) = [1 - \bar{b}(x)\Delta x]\left[(1 - \lambda\Delta x)p_k(x) + \sum_{i=1}^{k-1} p_i(x)\lambda g_{k-i}\Delta x\right] \quad (45)$$

where we assume $p_0(x) \equiv 0$. We can rewrite (45) as

$$\frac{p_k(x + \Delta x) - p_k(x)}{\Delta x} + [\lambda + \bar{b}(x)]p_k(x) = \lambda \sum_{i=1}^{k-1} p_i(x)g_{k-i} + o(\Delta x) \quad (46)$$

Taking the limit as $\Delta x \to 0$, we get the differential equation

$$\frac{d}{dx}p_k(x) + [\lambda + \bar{b}(x)]p_k(x) = \lambda \sum_{i=1}^{k-1} p_i(x)g_{k-i} \qquad k \geq 1. \quad (47)$$

Next, we consider the boundary conditions that p_0 and $p_k(x)$ require to satisfy. Noting that, in equilibrium, the rate at which L_t moves out of state 0 due to arrivals is equal to the rate at which it moves into state 0 from state 1 due to service completions, we have

$$\lambda p_0 = \int_0^\infty p_1(x)\bar{b}(x)dx. \quad (48)$$

Similarly, noting that the start of a new service follows either an arrival to an empty system or a service completion, we have

$$p_k(0) = \lambda g_k p_0 + \int_0^{\infty} p_{k+1}(x)\bar{b}(x)dx \qquad k \geq 1. \tag{49}$$

Finally, the normalization condition is given by

$$p_0 + \sum_{k=1}^{\infty} \int_0^{\infty} p_k(x)dx = 1. \tag{50}$$

3.3.2 Probability generating function of the distribution $p_k = \lim_{t \to \infty} Pr\{L_t = k\}$

The system of differential equations given (47) together with boundary conditions given in (48), (49) and the normalization condition given in (50) can be solved using transform methods as carried out in Appendix E to obtain the double transform $P^*(z, s)$ for $p_k(x)$

$$P^*(z, s) \equiv \sum_{k=1}^{\infty} z^k \int_0^{\infty} e^{-sx} p_k(x)dx$$
$$= \frac{\lambda(1-\rho)z[1-G(z)]}{B^*(\lambda - \lambda G(z)) - z} \frac{1 - B^*(s + \lambda - \lambda G(z))}{s + \lambda - \lambda G(z)}. \tag{51}$$

Noting that the probability that there are k customers present in the system at an arbitrary instant, p_k, for $k > 0$ can be derived from $p_k(x)$ by integrating it over x, namely

$$p_k = \int_0^{\infty} p_k(x)dx, \tag{52}$$

multiplying (52) by z^k and summing over $k > 0$ yield

$$\sum_{k=1}^{\infty} p_k z^k = P^*(z, s)\big|_{s=0}. \tag{53}$$

Observing that the generating function of the distribution for the number of customers present in the system at an arbitrary instant is given by

$$P(z) = p_0 + \sum_{k=1}^{\infty} p_k z^k \tag{54}$$

and substituting first (53) then (51) into (54), we obtain

$$P(z) = (1-\rho)\frac{(1-z)e^{-\lambda\bar{x}[1-G(z)]}}{e^{-\lambda\bar{x}[1-G(z)]} - z} \tag{55}$$

where we make use of

$$p_0 = 1 - \lambda\bar{x}g = 1 - \rho \tag{56}$$

as also derived in Appendix E. From (55) and (56), it is clear that p_k can be expressed as

$$p_k = p_0 h_k \tag{57}$$

where h_k satisfies

$$H(z) = \sum_{k=0}^{\infty} h_k z^k = \frac{(1-z)e^{-\lambda\bar{x}[1-G(z)]}}{e^{-\lambda\bar{x}[1-G(z)]} - z}. \tag{58}$$

Note that calculating p_k using the above transform results in the same p_k obtained by solving the systems of differential equations in (47) to (50) directly.

3.3.3 Departure Instants and Arbitrary Instants

Even though we have obtained the generating function of the equilibrium queue occupancy distribution at an arbitrary instant using the method of supplementary variables, the computational complexity of inverting the corresponding transform may become a formidable task. In order to facilitate an efficient calculation of the distribution, we next explore the relationship between the equilibrium queue occupancy distribution at a departure instant and that of an arbitrary instant.

The comparison of expressions given in (31) and (55) provides the relationship between the generating function of the equilibrium queue occupancy distribution at arbitrary instants and that of customer departure instants as

$$\begin{aligned} P(z) &= \frac{g(1-z)}{1-G(z)}Q^d(z) \\ &= B(z)Q^d(z) \end{aligned} \tag{59}$$

where $B(z)$ is known as the generating function of the backward recurrence time distribution with

$$B(z) = \sum_{k=0}^{\infty} b_k z^k = \frac{g(1-z)}{1-G(z)}. \tag{60}$$

By evaluating (59) at $z = 0$, we confirm the relationship

$$p_0 = g d_0 \tag{61}$$

as it may also be derived from equations (30) and (56). Further inverting the transforms on both sides of (59), p_k can be expressed as the convolution of two sequences, that is

$$\begin{aligned} p_k &= [b_k] \otimes [d_0 f_k] \\ &= (g d_0) \cdot \frac{1}{g}[b_k \otimes f_k] \\ &= p_0 h_k \end{aligned} \qquad (62)$$

where we make use of elementary convolution properties. The relationship given in (62) also provides an expression, as required in (57), for calculating h_k as

$$h_k = \frac{1}{g}[b_k \otimes f_k]. \qquad (63)$$

3.4 The Algorithm

In this section, we provide a summary of the results obtained in earlier sections such that the equilibrium distribution queue occupancy distribution can be calculated accurately and efficiently. For the sake of completeness, we rewrite the relevant expressions.

- $\{g_k^{(n)}\}$ - given in (14)

$$\begin{aligned} g_k^{(1)} &= g_k \\ g_k^{(n)} &= \sum_{i=1}^{k-1} g_i g_{k-i}^{(n-1)} \qquad n > 1 \end{aligned} \qquad (64)$$

- $\{\alpha_k\}$ - given in (10) and (15)

$$\alpha_0 = e^{-\lambda\bar{x}}$$

$$\alpha_k = \sum_{n=1}^{k} \frac{(\lambda\bar{x})^n}{n!} e^{-\lambda\bar{x}} g_k^{(n)} \qquad k \geq 1 \tag{65}$$

- $\{\beta_k\}$ - given in (18)

$$\beta_k = \sum_{m=1}^{k+1} g_m \alpha_{k-m+1} \qquad k \geq 0 \tag{66}$$

- $\{\hat{\alpha}_k\}$ and $\{\hat{\beta}_k\}$ - given in (39)

$$\hat{\alpha}_k = 1 - \sum_{i=0}^{k} \alpha_i \qquad \hat{\beta}_k = 1 - \sum_{i=0}^{k} \beta_i \qquad k \geq 0 \tag{67}$$

- $\{f_k\}$ - given in (40)

$$f_0 = 1$$

$$f_1 = \frac{1}{\alpha_0}\hat{\beta}_0$$

$$f_k = \frac{1}{\alpha_0}\left(\hat{\beta}_{k-1} + \sum_{i=1}^{k-1} \hat{\alpha}_{k-i} f_i\right) \qquad k \geq 2 \tag{68}$$

- $\{b_k\}$ - given in (60)

$$B(z) = \frac{g(1-z)}{1-G(z)} \tag{69}$$

At this step, we require to invert the transform on the right hand side of (69). As $G(z)$ is a polynomial in z, we can make use of partial fraction expansion on $B(z)$ and invert each fraction by inspection to calculate the sequence $\{b_k\}$. A more direct algorithm for numerical calculation of the same sequence is for further study.

- $\{h_k\}$ - given in (63)

$$h_k = \frac{1}{g}[b_k \otimes f_k] \qquad (70)$$

- p_0 - given in (56)

$$p_0 = 1 - \lambda\bar{x}g \qquad (71)$$

- $\{p_k\}$ - given in (62)

$$p_k = p_0 h_k \qquad (72)$$

3.5 Summary

In this chapter, we have established an algorithm to calculate the equilibrium queue occupancy distribution at an arbitrary instant of the $M^{[x]}/D/1/\infty$ queue, that is, the $M/D/1$ queue with group arrivals and unlimited buffer. We first use the method of imbedded Markov chain to obtain the distribution of number of cells left behind by a departing cell. Then, the method of supplementary variables is used to obtain the equilibrium probability distribution for the number of cells at an arbitrary time. Finally, we exploit the relationship between equilibrium probability distribution for the number of cells left behind by a departing cell and that of at an arbitrary time to develop an algorithm for calculating the required distribution.