CHAPTER 4
MODELLING OF THE PACKET DISCARD STRATEGY

In this chapter, we introduce the queueing system used to model the Packet Discard Strategy, namely the $M^{[x]}/D/1$ queue with threshold. With this queue, the whole packet is discarded upon arrival if the buffer fill exceeds a threshold value already. We will observe in the numerical results that modelling with group arrivals constitutes a worst case as compared to cell arrivals. We will also observe that having multiple input ports results in a queue length distribution very close to that of group arrivals.

Through the analysis, we determine the algorithm to calculate the equilibrium queue statistics. Note that when the queue length is under threshold, the $M^{[x]}/D/1$ queue with threshold behaves exactly like the $M^{[x]}/D/1/\infty$ queue. And in the analysis, we illustrate the relationship between these two queues which promotes efficient calculation of numerical results of the latter queue. In addition, we explore the relationships among traffic loading, probability of cell or packet loss, threshold value and effective utilization factor.

In section 4.1, we describe the $M^{[x]}/D/1$ queue with threshold. In section 4.2, the method of supplementary variables is used to obtain the equilibrium queue statistics at an arbitrary time. The relationship between the two queues are also established. Moreover, the queue statistics are derived. In Section 4.3, an algorithm is given to calculate the required distribution and queue statistics. Finally in section 4.4, we summarize the chapter.

4.1 Queue Description

The $M^{[x]}/D/1$ queue with Threshold is a variant of the $M^{[x]}/D/1$ queue in that the former incorporates a discarding mechanism based on the buffer fill. This discarding mechanism can be seen as a discarding-decision process that discards new groups upon arrivals whenever the queue length exceeds a threshold value. The queueing diagram representing the $M^{[x]}/D/1$ with threshold is shown in Figure 4.1. With the discarding-decision process, some arrivals are rejected. Hence we define λ_e to be the average rate of groups that are accepted into the queue and λ_{loss} to be the average discarding rate of customers where their sum is equal to λ.

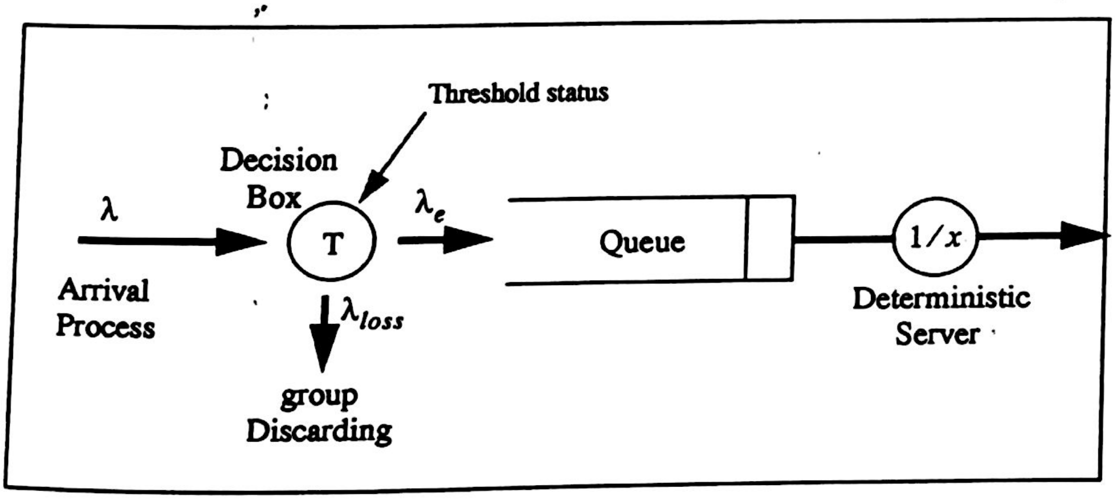

Figure 4.1: $M^{[x]}/D/1$ with Threshold and Discarding-decision

The function of the decision box is such that at each arrival instant, if the buffer fill is less than the threshold value, the decision box accepts the group, and if the buffer fill is greater than or equal to the threshold value, the decision box discards the group.

4.2 Analysis at Arbitrary Instants

In this section, we employ the method of supplementary variables and extend the derivation in section 3.3 to the case with threshold. We first define the stationary joint distribution for the number of customers present in the queue and the elapsed service time. We then derive the relations that the stationary distribution satisfies. The probability of the number of customers present in the queue at an arbitrary time is obtained by relating the derived relations to those in section 3.3.2. Finally, we determine the effective utilization factor and the probability of cell/packet loss.

4.2.1 Queue size and elapsed service time

As in Section 3.3.1, let us define the stationary joint distribution for the number of customers present in the queue and the elapsed service time as

$$p_0^T \equiv \lim_{t \to \infty} Pr\{L_t = 0\}, \tag{73}$$

$$p_k^T(x)dx \equiv \lim_{t \to \infty} Pr\{L_t = k, X_t^- \in x_{dx}\} \qquad k \geq 1, x \geq 0. \tag{74}$$

The $M^{[x]}/D/1$ queue with threshold operates in exactly the same manner as the $M^{[x]}/D/1/\infty$ queue when $T > k \geq 1$. Therefore, when we consider $p_k^T(x + \Delta x)$ for

$T > k \geq 1$, we have the same relationship between $p_k^T(x)$'s as the one for $p_k(x)$'s in (47) of Section 3.3.1. That is,

$$\frac{d}{dx}p_k^T(x) + [\lambda + \bar{b}(x)]p_k^T(x) = \lambda \sum_{i=1}^{k-1} p_i^T(x) g_{k-i} \qquad T > k \geq 1. \tag{75}$$

Next, we consider $p_k^T(x + \Delta x)$ for $k \geq T$. The event $\{L_{t+\Delta x} = k, X^-_{t+\Delta x} \in (x + \Delta x)_{dx}\}$ occurs either when $\{L_t = k, X^-_t \in x_{dx}\}$ and there are no arrivals during Δx, or when $\{L_t = k - i | k - i < T, X^-_t \in x_{dx}\}$ and there is an arrival of size i (the one that triggered the PDS) during Δx. A further requirement in either case is that there are no service completions during Δx. Thus, we have

$$p_k^T(x + \Delta x) = [1 - \bar{b}(x)\Delta x]\left[p_k^T(x) + \sum_{i=1}^{T-1} p_i^T(x)\lambda g_{k-i}\Delta x\right] \tag{76}$$

where we assume $p_0^T(x) \equiv 0$. We can rewrite (76) as

$$\frac{p_k^T(x + \Delta x) - p_k^T(x)}{\Delta x} + \bar{b}(x)p_k^T(x) = \lambda \sum_{i=1}^{T-1} p_i^T(x) g_{k-} \tag{77}$$

Taking the limit as $\Delta x \to 0$, we get the differential equation

$$\frac{d}{dx}p_k^T(x) + \bar{b}(x)p_k^T(x) = \lambda \sum_{i=1}^{T-1} p_i^T(x) g_{k-i} \qquad k \geq T. \tag{78}$$

Now, let us consider the boundary conditions that p_0^T and $p_k^T(x)$ require to satisfy. Again, we have the same relationship between $p_1^T(x)$ and p_0^T as the one between $p_1(x)$ and p_0 in (48). That is,

$$\lambda p_0^T = \int_0^\infty p_1^T(x)\bar{b}(x)dx. \tag{79}$$

Noting that the start of a new service follows either an arrival to an empty system or a service completion regardless of the queue size, hence we have the same relationship between $p_k^T(0)$'s and that between $p_k(0)$'s in (49). That is,

$$p_k^T(0) = \lambda g_k p_0^T + \int_0^\infty p_{k+1}^T(x)\bar{b}(x)dx \qquad 1 \le k \le \infty. \tag{80}$$

Finally, we have the same normalization condition as in (50) and is given by

$$p_0^T + \sum_{k=1}^{\infty} \int_0^\infty p_k^T(x)dx = 1. \tag{81}$$

4.2.2 *The $M^{[x]}/D/1/\infty$* queue and the *$M^{[x]}/D/1$* queue with threshold

In solving the differential equations of the queue, we consider the case for $1 \le k < T$ and the case for $k \ge T$ separately. As the differential equations and boundary conditions of $p_k^T(x)$ and those of $p_k(x)$ in section 3.3.1 are the same for $1 \le k < T$, hence if we solve the systems of differential equations iteratively, we will obtain the same ratio for $\frac{p_k^T}{p_0^T}$ and $\frac{p_k}{p_0}$ for $1 \le k < T$. That is,

$$\frac{p_k^T}{p_0^T} = \frac{p_k}{p_0} = h_k \qquad \text{for } 1 \le k < T \tag{82}$$

where p_k^T is the equilibrium probability of the number of customers in the system at an arbitrary instant. Therefore, we can make use of the numerical calculation of h_k from section 3.3 to calculate p_k^T, if p_0^T is known. Fortunately, p_0^T can be derived from the relationship between $p_k^T(0)$ and the probability that the server is busy. As derived in Appendix F, we have

$$p_0^T = \left[1 + \lambda \bar{x} g \sum_{k=0}^{T-1} h_k\right]^{-1}. \tag{83}$$

For the case where $k \geq T$, it can be seen that the maximum possible number of customers in the system is $T - 1 + G_{max}$ where G_{max} is the maximum possible number of customers within a group. Hence,

$$p_k^T = 0 \qquad \text{for } k > (T - 1 + G_{max}) \tag{84}$$

The equilibrium probability of having T to $T - 1 + G_{max}$ customers in the system at an arbitrary time is not of practical interest as the probability is not required for the calculation of the queue performance.

4.2.3 Queue performance

In this section, we establish the performance measures for the $M^{[x]}/D/1$ queue with threshold. These measures include effective utilization factor and the probability of losing a group due to the discarding mechanism.

The probability of losing a group is the probability of a group of customers arrives and sees the system is not accepting any new arrivals. Our queue will not accept new arrivals if there are T or more customers in the queue at the arrival instant. Therefore,

$$P[k \geq T] \text{at arrival instants} = P[\text{the system accepts no new arrivals}]$$

By the PASTA (Poisson Arrivals See Time Averages) property, when a new group of customers arrives, the probability of the group finding the system is accepting no new arrivals is equal to the equilibrium probability of having T or more customers in the queue at arrival instants. Hence,

$$P_{loss}^{T} = \sum_{k=T}^{\infty} p_k^T = 1 - p_0^T \sum_{k=0}^{T-1} h_k. \tag{85}$$

By making use of (83), we have

$$P_{loss}^{T} = 1 - \frac{\sum_{k=0}^{T-1} h_k}{1 + \lambda \bar{x} g \sum_{k=0}^{T-1} h_k}. \tag{86}$$

The rate of loss of groups, λ_{loss}^{T}, is the product of the probability of losing groups, P_{loss}^{T} and the average arrival rate of groups. That is,

$$\lambda_{loss}^{T} = \lambda P_{loss}^{T}. \tag{87}$$

In the queueing theory literature, the utilization factor, ρ, of a queueing system is defined as the fraction of time the server is busy. Therefore,

$$\rho = \frac{\text{average arrival rate of customers}}{\text{average service rate of customers}}$$

$$= \text{average arrival rate of customers} \times \text{average service time.}$$

For example, for the $M/D/1/\infty$ queue, we have

$$\rho = \lambda\bar{x}, \tag{88}$$

and for the $M^{[x]}/D/1/\infty$ queue, we have

$$\rho = \lambda\bar{x}g. \tag{89}$$

However, for a system with loss, not all of the arrived customers are serviced. Some of them are discarded / lost. Therefore,

average arrival rate of customers × average service time
≠ the fraction of time the server is busy.

In analysing systems with loss, we use the term effective utilization factor, ρ_e, to represent the fraction of time the server is busy. Hence,

ρ_e = average arrival rate of customers that are not discarded × average service time

and for the $M^{[x]}/D/1$ queue, we have

$$\rho_e = (\lambda - \lambda_{loss})\bar{x}g,$$

where λ_{loss} is the average discarding rate of customers. But we still use the term

$$\rho = \lambda\bar{x}g$$

as a measurement of the level of loading of the system, namely offered load.

For the $M^{[x]}/D/1$ queue with threshold, the average arrival rate of customers is the product of the average arrival rate of groups and the average length of groups. Hence,

$$\rho_e^T = (\lambda - \lambda_{loss}^T)\bar{x}g. \tag{90}$$

$$\rho^T = \lambda\bar{x}g. \tag{91}$$

As the definition of the effective utilization factor, ρ_e^T, is the fraction of the time the system is busy and p_0^T is the equilibrium probability of the system being empty at an arbitrary instant, we have

$$\rho_e^T = 1 - p_0^T. \tag{92}$$

By make use of (83), the effective utilization factor can be expressed as

$$\rho_e^T = \frac{\lambda\bar{x}g\sum_{k=0}^{T-1} h_k}{1 + \lambda\bar{x}g\sum_{k=0}^{T-1} h_k}. \tag{93}$$

By substituting (93) into (90), we have the same expression for P_{loss}^T as derived in (85).

4.3 The Algorithm

In this section, we provide a summary of the results obtained in section 4.2 such that the equilibrium queue statistics can be calculated accurately and efficiently. For the sake of completeness, we rewrite the relevant expressions.

- $\{h_k\}$ - given in (63) in section 3.3.3

$$h_k = \frac{1}{g}[b_k \otimes f_k] \tag{94}$$

- p_0^T - given in (83)

$$p_0^T = \left[1 + \lambda\bar{x}g \sum_{k=0}^{T-1} h_k^{\infty}\right]^{-1} \tag{95}$$

- $\{p_k^T\}$ - given in (82)

$$p_k^T = p_0^T h_k \quad 1 \le k < T \tag{96}$$

- P_{loss}^T - given in (86)

$$P_{loss}^T = 1 - \frac{\sum_{k=0}^{T-1} h_k}{1 + \lambda\bar{x}g \sum_{k=0}^{T-1} h_k} \tag{97}$$

- ρ^T - given in (91)

$$\rho^T = \lambda\bar{x}g \tag{98}$$

- ρ^T - given in (92)

$$\rho_e^T = 1 - p_0^T \tag{99}$$

4.4 Summary and Discussions

In this chapter, we established an algorithm to calculate the equilibrium queue occupancy distribution at an arbitrary instant of the $M^{[x]}/D/1$ queue with threshold. The method of supplementary variables is used to obtain the equilibrium probability distribution for the number of cells at an arbitrary time. In the analysis, we exploit the relationship of equilibrium probability distribution between the $M^{[x]}/D/1/\infty$ queue and the $M^{[x]}/D/1$ queue with threshold to develop an algorithm for calculating the required distribution and queue statistics efficiently.

The $M^{[x]}/D/1$ queue with threshold operates in exactly the same way as the $M^{[x]}/D/1/\infty$ queue up to a particular level of buffer fill. Therefore, it can be considered as a variant of the $M^{[x]}/D/1/\infty$ queue in which the dynamics of the queue occupancy changes when a particular value of queue occupancy is reached. This similarity allows us to make use of h_k to promote efficient calculation of equilibrium queue statistics for the $M^{[x]}/D/1$ queue with threshold. The discarding mechanism introduce some loss into the queue. The loss will affect the percentage of the time the server is busy, hence affecting the probability of having an empty queue and finally affecting the probability of the rest of the queue occupancy.

The algorithm to solve for the equilibrium queue statistics for both queues is to identify the five fundamental equations which characterize the behaviour of a queue. These equations are listed in Table 4.1. Note that with group arrivals, we have $\rho = \lambda g x$.

The first equation is the relationship between p_k and p_0. Since both queues operate in the same way up to a particular level of buffer fill, the relations are the same up to a particular value. Here, the function h_k can be considered as the dynamics of the queue. The second equation is the normalization condition. The third equation states the probability of loss. The derivation of this equation requires understanding of how a variant of the queue operates. The fourth equation states the relationship between the probability of the system being busy and the probability of loss. The last equation states that the probability of the system being busy is equal to one minus the probability of having an empty system. Once the probability of having an empty

$M^{[x]}/D/1/\infty$	$M^{[x]}/D/1$ with Threshold
$p_k = p_0 h_k$	$p_k^T = p_0^T h_k$ for $k < T$
$\sum_{k=0}^{\infty} p_k^T = 1$	$\sum_{k=T}^{\infty} p_k^T = 1 - p_0^T \sum_{k=0}^{T-1} h_k$
$\rho_e = (1 - P_{loss})\rho$	$\rho_e^T = (1 - P_{loss}^T)\rho$
$P_{loss} = 0$	$P_{loss}^T = 1 - p_0^T \sum_{k=0}^{T-1} h_k$
$\rho_e \equiv 1 - p_0$	$\rho_e^T \equiv 1 - p_0^T$

Table 4.1: The Five equations which Characterize the Behaviour of a Queue.

queue is derived, the queue statistics can be calculated according the rest of the equations.

CHAPTER 5
MODELLING OF THE CELL DISCARD STRATEGY

In this chapter, we introduce another variant of the $M^{[x]}/D/1$ queue to model the Cell Discard Strategy, namely the $M^{[x]}/D/1/B$ queue where B denotes the buffer size. With this queue, upon a group arrival the queue will discard the cells within a group that cannot fit into the remaining space in the queue. Therefore, an arriving group of customers may be totally accepted, partially accepted into the queue or totally rejected. This discarding nature is similar to the Cell Discard Strategy operating within an ATM switch. With the queueing model, we use the goodput to investigate the performance of the Cell Discard Strategy.

Through the analysis, we determine the algorithm to calculate the equilibrium queue statistics. Note that as in the case of the $M^{[x]}/D/1$ queue with threshold, the $M^{[x]}/D/1/B$ queue operates exactly as the $M^{[x]}/D/1/\infty$ queue before the queue is full. And in the analysis, we illustrate the relationship between the $M^{[x]}/D/1/\infty$ queue and the $M^{[x]}/D/1/B$ queue. Moreover, we explore the relationships among traffic loading, probability of cell/packet loss, buffer size and effective utilization factor.

In section 5.1, we describe the $M^{[x]}/D/1/B$ queue. In section 5.2, the method of supplementary variables is used to obtain the equilibrium queue statistics at an arbitrary time. The relationship between the two queues is also established. In addition, the queue statistics are derived In Section 5.3, an algorithm is given to calculate the required distributions and queue statistics. Finally in Section 5.4, we summarize the chapter.

5.1 Queue Description

The $M^{[x]}/D/1/B$ queue is a variant of the $M^{[x]}/D/1$ queue in that the former has a buffer of limited size B. The queueing diagram representing the $M^{[x]}/D/1/B$ queue is shown in Figure 5.1. When a group arrives, the system keeps accepting into the buffer the cells in the group until the buffer size reaches B. Then, the rest of the cells in the group are discarded. As a result, the system discards cells at an average rate λ_{loss}. We use this discarding-process to model the Cell Discard Strategy.

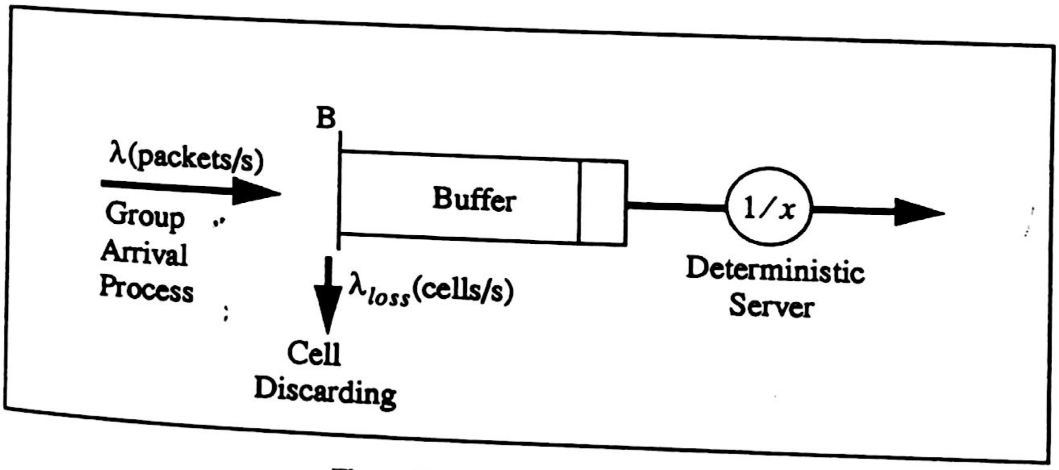

Figure 5.1: The $M^{[x]}/D/1/B$ queue

5.2 Analysis at Arbitrary Instants

In this section, we employ the method of supplementary variables and extend the derivation in section 3.3 to the case with limited buffer. We first define the stationary joint distribution for the number of customers present in the queue and the elapsed service time. Next, we derive the relations that the stationary distribution satisfies.

The probability of the number of customers present in the queue at an arbitrary time is obtained by relating the derived relations to those in section 3.3.2. Finally, we determine the effective utilization factor, probability of cell loss and the probability of packet loss.

5.2.1 Queue size and elapsed service time

As in Section 3.3.1, let us define the stationary joint distribution for the number of customers present in the queue and the elapsed service time as

$$p_0^B \equiv \lim_{t \to \infty} Pr\{L_t = 0\}, \tag{100}$$

$$p_k^B(x)dx \equiv \lim_{t \to \infty} Pr\{L_t = k, X_t^- \in x_{dx}\} \qquad k \geq 1, x \geq 0. \tag{101}$$

As the $M^{[x]}/D/1/B$ queue operates in exactly the same manner as the $M^{[x]}/D/1/\infty$ queue when $k < B$, we have the same relationship between $p_k^B(x)$'s and the one for $p_k(x)$'s in (47) of Section 3.3.1. That is,

$$\frac{d}{dx}p_k^B(x) + [\lambda + \bar{b}(x)]p_k^B(x) = \lambda \sum_{i=1}^{k-1} p_i^B(x) g_{k-i} \qquad 1 \leq k < B. \tag{102}$$

Next, we consider $p_k^B(x + \Delta x)$ for $k = B$. Note that $p_k^B(x) = 0$ for $k > B$. The event $\{L_{t+\Delta x} = B, X_{t+\Delta x}^- \in (x + \Delta x)_{dx}\}$ occurs either when $\{L_t = B, X_t^- \in x_{dx}\}$ and there are some or no arrivals during Δx, or when $\{L_t = B - i, X_t^- \in x_{dx}\}$ and there is an arrival of size greater than to equal to i (the group that fill up the buffer) during

Δx. A further requirement in either case is that there are no service completions during Δx. Thus, we have

$$p_B^B(x+\Delta x) = [1-\bar{b}(x)\Delta x]\left[p_B^B(x) + \sum_{j=B}^{\infty}\sum_{i=1}^{B-1} p_i^B(x)g_{j-i}\lambda\Delta x\right]. \tag{103}$$

where we assume $p_0^B(x) \equiv 0$. We can rewrite (103) as

$$\frac{p_B^B(x+\Delta x) - p_B^B(x)}{\Delta x} + \bar{b}(x)p_B^B(x) = \lambda\sum_{j=B}^{\infty}\sum_{i=1}^{B-1} p_i^B(x)g_{j-i} + O(\Delta x).$$

Taking the limit as $\Delta x \to 0$, we get the differential equation

$$\frac{d}{dx}p_B^B(x) + \bar{b}(x)p_B^B(x) = \lambda\sum_{j=B}^{\infty}\sum_{i=1}^{B-1} p_i^B(x)g_{j-i} \tag{104}$$

Now, let us consider the boundary conditions that p_0^B and $p_k^B(x)$ require to satisfy. Again, we have the same relationship between $p_1^B(x)$ and p_0^B and the one between $p_1(x)$ and p_0 in (48). That is,

$$\lambda p_0^B = \int_0^{\infty} p_1^B(x)\bar{b}(x)dx. \tag{105}$$

Noting that the start of a new service follows either an arrival to an empty system or a service completion, and that $p_B^B(0)$ cannot occur at customer departure instants, we have

$$p_k^B(0) = \lambda g_k p_0^B + \int_0^\infty p_{k+1}^B(x)\bar{b}(x)dx \qquad 1 \le k < B. \tag{106}$$

$$p_B^B(0) = \sum_{i=B}^{\infty} \lambda g_i p_0^B. \tag{107}$$

Finally, we have the same normalization condition as in (50) and is given by

$$p_0^B + \sum_{k=1}^{B} \int_0^\infty p_k^B(x)dx = 1. \tag{108}$$

5.2.2 $M^{[x]}/D/1/\infty$ queue and $M^{[x]}/D/1/B$ queue

In solving the differential equations of the queue, we consider the case for $1 \le k < B$ and the case for $k = B$ separately. As the differential equations and boundary conditions of $p_k^B(x)$ and $p_0^B(x)$ in section 5.2.1 and those of $p_k(x)$ and p_0 in section 3.3.1 are the same for $1 \le k < B$. Hence, if we solve the systems of differential equations iteratively, we will obtain the same ratio for $\frac{p_k}{p_0^B}$ and $\frac{p_k}{p_0}$ for $1 \le k < B$. That is,

$$\frac{p_k^B}{p_0^B} = \frac{p_k}{p_0} = h_k \qquad \text{for } 1 \le k < B \tag{109}$$

where p_k^B is the equilibrium probability of the number of customers in the system at an arbitrary instant. Hence, we can make use of the results of h_k from section 3.3 to calculate p_k^B, if p_0^B is known. Fortunately, p_0^B can be derived from the relationship between $p_k^B(0)$ and the probability that the server is busy. As derived in Appendix G, we have

$$p_0^B = \left[1 + \lambda\bar{x}g\sum_{k=0}^{B-1} h_k\left(1 - \sum_{i=B+1-k}^{\infty} g_i\frac{i-(B-k)}{g}\right)\right]^{-1} \tag{110}$$

For the case when $k = B$, we can make use of the normalization condition and (110) to obtain

$$p_B^B = 1 - \sum_{k=0}^{B-1} p_k^B = 1 - \frac{\sum_{k=0}^{B-1} h_k^{\infty}}{\left[1 + \lambda\bar{x}g\sum_{k=0}^{B-1} h_k\left(1 - \sum_{i=B+1-k}^{\infty} g_i\frac{i-(B-k)}{g}\right)\right]} \tag{111}$$

Hence, we can make use of the results of h_k from section 3.3 to calculate p_k^B is known.

5.2.3 Queue performance

Now, we derive the expression for effective utilization factor for cells, ρ_e^B, and the effective utilization factor of packets, that is, the goodput of the system. Let us first examine the situation of overflow which results in cell loss. The buffer overflows when the size of an arriving group is larger than the size of the available buffer space. When this occurs, some or all of the cells of the arriving group are lost. Note that the probability that a tagged cell belongs to a packet of size i is ig_i/g. Hence the probability of a tagged cell belongs to the proportion of the cells that are discarded in a packet of size i is $\frac{i-(B-k)}{g}g_i$ where $B-k$ is the remaining buffer size. Hence, we have

$$P^B_{c\_loss} = \sum_{k=0}^{B} p^B_k \sum_{i=B+1-k}^{\infty} \frac{i-(B-k)}{g}g_i. \tag{112}$$

By making use of (109) and (110), we have

$$P^B_{c\_loss} = 1 - \frac{\sum_{k=0}^{B-1} h_k\left(1 - \sum_{i=B+1-k}^{\infty} \frac{i-(B-k)}{g}g_i\right)}{1 + \lambda\bar{x}g\sum_{k=0}^{B-1} h_k\left(1 - \sum_{i=B+1-k}^{\infty} \frac{i-(B-k)}{g}g_i\right)}. \tag{113}$$

We also have the rate of loss of cells, λ^B_{loss} equal to the product of the probability of losing cells and the average arrival rate of cells. That is,

$$\lambda^B_{loss} = \lambda P^B_{c\_loss}. \tag{114}$$

Recall from section 4.2.2 that the definition of the effective utilization factor, ρ_e, is the fraction of the time the system is busy. And p^B_0 is the equilibrium probability of the system being empty at an arbitrary instant. Hence,

$$\rho^B_e = 1 - p^B_0. \tag{115}$$

Since there is loss in this queue, we have

$$\rho^B_e = (\lambda - \lambda^B_{loss})\bar{x}g. \tag{116}$$

Note that the offered load of the system is

$$\rho^B = \lambda \bar{x} g . \qquad (117)$$

By making use of (110) and (115), we have the effective utilization factor for cells as

$$\rho_e^B = \frac{\lambda \bar{x} g \sum_{k=0}^{B-1} h_k \left(1 - \sum_{i=B+1-k}^{\infty} \frac{i-(B-k)}{g} g_i \right)}{1 + \lambda \bar{x} g \sum_{k=0}^{B-1} h_k \left(1 - \sum_{i=B+1-k}^{\infty} \frac{i-(B-k)}{g} g_i \right)} . \qquad (118)$$

Here, there is a need to distinguish the probability of cell loss, the probability of a packet being corrupted or dead, and the probability of a cell being dead, denoted as $P^B_{c\_loss}$, $P^B_{p\_dead}$ and $P^B_{c\_dead}$ respectively. $P^B_{c\_loss}$ is the probability of a cell being discarded due to buffer overflow. $P^B_{p\_dead}$ is the probability of a packet being corrupted, that is, being totally destroyed or partly destroyed due to buffer overflow. And $P^B_{c\_dead}$ is the probability of a cell being useless because the packet which it belongs to is corrupted. By definition, the effective utilization factor, ρ_e^B, that we are considering is the throughput of cells of the system. Hence, it depends on $P^B_{c\_loss}$ rather than $P^B_{p\_loss}$.

Next, let us consider the goodput of packets of the system. By making use of the PASTA property, we have

P[a packet going through the queue without being corrupted]

= P[all the cells of an arriving group will be accepted by the queue]

= P[there is enough space in the buffer to accept the whole group of cells]

$$= \sum_{k=0}^{B} \text{P[the arriving group sees the queue having } k \text{ customers in it] x P[the size of the group } \le B-k]$$

$$\Rightarrow P^B_{p\_loss} = 1 - \sum_{k=0}^{B} p^B_k \sum_{i=1}^{B-k} g_i = 1 - \sum_{k=0}^{B-1} p^B_k \sum_{i=1}^{B-k} g_i \qquad (119)$$

Therefore, the goodput of packets, $\rho^B_{g\_packet}$, is

$$\rho^B_{g\_packet} = \lambda(1 - P^B_{p\_loss})\bar{x}g \qquad (120)$$

By making use of (109), (110) and (119), we get

$$\rho^B_{g\_packet} = \frac{\lambda\bar{x}g \sum_{k=0}^{B-1} h_k \sum_{i=1}^{B-k} g_i}{1 + \lambda\bar{x}g \sum_{k=0}^{B-1} h_k \left(1 - \sum_{i=B+1-k}^{\infty} \frac{i-(B-k)}{g} g_i\right)} . \qquad (121)$$

Although ρ_{g_packet} shows the goodput of packets, it may be misleading when offered load is large with multi-modal packet size distributions. When offered load is high, most of the corrupted packets would be large packets and hence the goodput would stay high. In order to obtain the true goodput of the queue, we should consider the goodput of cells. That is, the cells belong to good packets. We denote this goodput of cells as $\rho^B_{g\_good\_cells}$.

In order to obtain the goodput of cells, we consider the probability of a cell being dead, $P^B_{c\_dead}$, due to the packet to which it belongs being corrupted. By making use of the PASTA property again, we have

$(1 - P^B_{c\_dead})$

=P[a cell accepted by the queue and the packet to which the cell belong stays good]

= $\sum\sum$ P[a cell belonging to a packet of size i arrives and sees the queue having k customers in it and the whole packet can be accepted into the queue]

= $\sum_k\sum_i$ P[there are k customers in the queue when the queue is in equilibrium] x P[a cell belonging to a packet of size i] x P[a packet of size i being accepted into the queue totally | the queue has k customers in it]

The probabilities of the last expression are p^B_k, $\frac{ig_i}{g}$ and $P[accept \mid k]$ respectively where

$$P[accept \mid k] = \begin{cases} 1 & i \le B-k \\ 0 & i > B-k \end{cases}.$$

Therefore, we get

$$P^B_{c\_dead} = 1 - \sum_{k=0}^{B} p^B_k \sum_{i=1}^{\infty} \frac{ig_i}{g} P[accept \mid k]$$

$$= 1 - \frac{\sum_{k=0}^{B} h_k \sum_{i=1}^{B-k} \frac{ig_i}{g}}{\left[1 + \lambda \bar{x} g \sum_{k=0}^{B-1} h_k \left(1 - \sum_{i=B+1-k}^{\infty} g_i \frac{i-(B-k)}{g}\right)\right]}, \qquad (122)$$

where we make use of (110)

And we have

$$\rho^{B}_{g\_good\_cells} = \lambda(1 - P^{B}_{c\_dead})\bar{x}g \quad . \tag{123}$$

By making use of (122), we obtain

$$\rho^{B}_{g\_good\_cells} = \frac{\lambda\bar{x}g \sum_{k=0}^{B-1} h_k \sum_{i=1}^{B-k} \frac{ig_i}{g}}{1 + \lambda\bar{x}g \sum_{k=0}^{B-1} h_k \left(1 - \sum_{i=B+1-k}^{\infty} \frac{i-(B-k)}{g} g_i\right)} . \tag{124}$$

5.3 The Algorithm

In this section, we provide a summary of the results obtained in section 5.2 such that the equilibrium queue statistics can be calculated accurately and efficiently. For the sake of completeness, we rewrite the relevant expressions.

- $\{h_k\}$ - given in (63) in section 3.3.3

$$h_k = \frac{1}{g}[b_k \otimes f_k]. \tag{125}$$

- p_0^B - given in (110)

$$p_0^B = \left[1 + \lambda\bar{x}g \sum_{k=0}^{B-1} h_k \left(1 - \sum_{i=B+1-k}^{\infty} \frac{i-(B-k)}{g} g_i\right)\right]^{-1} . \tag{126}$$

- $\{p_k^B\}$ - given in (109)

$$p_k^B = p_0^B h_k \qquad k < T . \tag{127}$$

- $P^B_{c\_loss}$ - given in (113)

$$P^B_{c\_loss} = 1 - \frac{\sum_{k=0}^{B-1} h_k \left(1 - \sum_{i=B+1-k}^{\infty} \frac{i-(B-k)}{g} g_i\right)}{1 + \lambda \bar{x} g \sum_{k=0}^{B-1} h_k \left(1 - \sum_{i=B+1-k}^{\infty} \frac{i-(B-k)}{g} g_i\right)}. \tag{128}$$

- ρ^B_e - given in (115)

$$\rho^B_e = 1 - p^B_0. \tag{129}$$

- $P^B_{p\_loss}$ - given in (119)

$$P^B_{p\_loss} = 1 - \sum_{k=0}^{B} p^B_k \sum_{i=1}^{B-k} g_i = 1 - p^B_0 \sum_{k=0}^{B-1} h_k \sum_{i=1}^{B-k} g_i. \tag{130}$$

- $\rho^B_{g\_packet}$ - given in (120)

$$\rho^B_{g\_packet} = \lambda(1 - P^B_{p\_loss})\bar{x}g. \tag{131}$$

- $P^B_{c\_dead}$ - given in (122)

$$P^B_{c\_dead} = 1 - \frac{\sum_{k=0}^{B} h_k \sum_{i=1}^{B-k} \frac{i g_i}{g}}{\left[1 + \lambda \bar{x} g \sum_{k=0}^{B-1} h_k \left(1 - \sum_{i=B+1-k}^{\infty} g_i \frac{i-(B-k)}{g}\right)\right]}. \tag{132}$$

- $\rho^B_{g\_good\_cells}$ - given in (123)

$$\rho^B_{g\_good\_cells} = \lambda(1 - P^B_{c\_dead})\bar{x}g. \tag{133}$$

5.4 Summary and Discussions

In this chapter, we established an algorithm to calculate the equilibrium queue statistics at an arbitrary instant of the $M^{[x]}/D/1/B$ queue, that is, the $M/D/1$

queue with group arrivals and buffer of limited size B. We use the method of supplementary variables to obtain the equilibrium queue statistics for the number of cells at an arbitrary time. We exploit the relationship of equilibrium probability distribution between the $M^{[x]}/D/1/\infty$ queue and the $M^{[x]}/D/1/B$ queue to develop an algorithm for calculating the required distribution and queue statistics.

A comparison of the five fundamental equations which characterize the behaviour of a queue are listed in Table 5.1. Note that with group arrivals, we have $\rho = \lambda\bar{x}g$. The

$M^{[x]}/D/1/\infty$	$M^{[x]}/D/1/B$
$p_k = p_0 h_k$	$p_k^B = p_0^B h_k \quad \text{for } k < B$
$\sum_{k=0}^{\infty} p_k^T = 1$	$p_B^B = 1 - p_0^B \sum_{k=0}^{B-1} h_k$
$\rho_e = (1 - P_{loss})\rho$	$\rho_e^B = (1 - P_{loss}^B)\rho$
$P_{loss} = 0$	$P_{loss}^B = p_0^B \sum_{k=0}^{B-1} h_k \sum_{i=B+1-k}^{\infty} \frac{i-(B-k)}{g} g_i$
$\rho_e \equiv 1 - p_0$	$\rho_e^B \equiv 1 - p_0^B$

Table 5.1: The Five equations which Characterize the Behaviour of a Queue.

two variants we considered in the thesis both introduce some loss into the queue. The loss will affect the percentage of the time the server is busy, hence affecting the probability of having an empty queue and finally affecting the probability of the rest of the queue occupancy.

We believe that the technique in relating a queueing system with its variants can be applied to other kind of queues, other than the $M^{[x]}/D/1$ queue, with no or minor modifications but is subject to further investigations. Note that this technique applies only to a variant of a queue that either changes the arrival or service process when a level of buffer fill is exceeded.