# CHAPTER 7

# CONCLUSION

This thesis reports on the performance of a Packet Discard Strategy (PDS) used as the basis for congestion control of data traffic associated with Unspecified Bit Rate transfer service in ATM data networks. Both analytical models and simulations were employed to investigate the performance. Our mathematical analysis used the $M^{[x]}/D/1$ queue with threshold for modelling the ATM switch operating with the PDS. For the switch operating without the PDS, the $M^{[x]}/D/1/\infty$ queue was employed. For comparison, we used the $M^{[x]}/D/1/B$ queue to model the performance of an ATM switch operating with a cell rather than a packet discard strategy. Simulations were used to study the behaviour of an ATM switch operating with and without the PDS. In order to compare the results from the analytical model with those from the simulations, both the analytical model and the simulations were set to generate packets according to the Poisson arrival process. However for the simulations, the packets generated are segmented into cells and are interleaved with other packets through the switching action before the cells are delivered to an output buffer. This represents a level of cell multiplexing at the input ports of the switch. On the other hand with the analytical model, the packets generated are delivered to an output buffer directly as a group of cells without any interleaving with other packets. The absence of interleaving in the analytical model is for tractability.

Using the analytical model and the simulations, we first studied the behaviour of an ATM switch output buffer operating without any discard. The numerical results show that the buffer occupancy distribution obtained from the analytical model is a good approximation to the real behaviour as considered in the simulation. In fact, the buffer occupancy distribution obtained from the analytical model is slightly higher than the one obtained from the simulations. Hence the results from the analytical model represent a conservative estimation of the performance of the output buffer. Results show that the accuracy of the group arrival analytical model improves with the traffic intensity as well as with the size of the switch in terms of the number of input ports contributing traffic. We also observed that the accuracy improves as the rate of the server or output link decreases. In our queueing model, as long as the traffic intensity is unchanged, changing the ratio between the arrival and service rate does not affect the output buffer occupancy distribution. However as observed in the simulations, the change of this ratio does affect the distribution.

With both the analytical model and the simulations, we also studied the behaviour of an output buffer operating with the PDS. Numerical results from both the analytical model and the simulations show that the packet loss probability of an output buffer decreases as the threshold increases. The results also show that the goodput characteristic of an output port is close to the ideal goodput characteristic. Again, results from the analytical model and the simulations are close to each other. The packet loss probability from the analytical model is higher than the one from the simulations, and the goodput characteristic from the analytical model is lower than the one from the simulations. The results from the analytical model again represents a conservative estimation of the packet loss probability and the goodput of an output buffer when PDS is in operation.

With the closeness of the results from the analytical model and simulations in terms of buffer occupancy distribution and packet loss probability, it follows that the analytical model based on group arrivals is a useful tool for design. From our numerical results for a particular traffic pattern, a threshold value can be chosen for Markovian traffic to achieve an acceptable operational level of performance. For example, with MAN like packet size distribution, 80% traffic intensity and $10^{-2}$ packet loss probability, our estimated threshold value is 500 cells. With LAN like packet size distribution and the same traffic intensity, having such a threshold can achieve a much better packet loss probability of $3.1 \times 10^{-5}$.

With the queuing analysis, we studied the parameters affecting the performance of the PDS and the Cell Discard Strategy. With the PDS, increasing the threshold value results in performance close to the ideal goodput characteristics. However, with the Cell Discard Strategy, increasing the buffer size only results in better performance in the underload region and once the link becomes overloaded, the performance degrades irrespective of the buffer size. We also compared the performance of the PDS with that of the Cell Discard Strategy using a small threshold value and a large buffer size to show the outstanding performance of the PDS. It can be concluded that an ATM switch operating with the PDS performs much better than that with the Cell Discard Strategy. With the PDS, the cell loss multiplication is largely reduced and hence the system maintains 100% goodput even under sustained overload. Even though the goodput of the system with the Cell Discard Strategy doesn't drop to zero immediately once it is overloaded, it should be noted that most of the packets that go through without being corrupted are small packets. On the other hand, the systems with the PDS doesn't bias towards small packets at all. It maintains on the output the same packet size distribution as the packets come.

Through the analysis of the $M^{[x]}/D/1$ queueing system and its variants, we explored the relationship between buffer occupancy at departure instants and that at arbitrary instants. Moreover, the relationships between the queueing system and each of its variants were also investigated. These two relationships lead to effective numerical calculations of queue statistics of each of the queues. We believe that these kinds of relationships exist between other queues and their respective variants as well, however, this is left for further study and is outside the scope of this thesis.

In this thesis, we consider applying the PDS to an ATM switch. However, it is useful to apply the PDS to anywhere in ATM data networks associated with UBR traffic where buffer may overflow and hence cell loss multiplication is threatened. An example is a slow TE connected to a high speed concentrator in ATM networks. With the UBR service, the traffic intensity may be changing from time to time in an ATM data network, instead of using a fixed threshold, the threshold value can be chosen dynamically by periodically estimating the incoming traffic intensity.

In order to prevent buffer overflow when the Packet Discard Strategy is in operation, the proportion of the buffer above the threshold should be large enough to accommodate the packets in transit at the time the decision was taken to reject new packets. Otherwise buffer overflows above threshold will occur and the goodput performance of the PDS will degrade. Our simulations show that even with buffer overflows above threshold, the goodput performance of the Packet Discard Strategy is still far better than that of the Cell Discard Strategy, but it is the subject of ongoing work. The queueing model considered in this thesis provides a performance comparison between the Packet Discard Strategy and the Cell Discard Strategy. It also provides a reasonable basis for design in determining the threshold value for a desired performance. However, the proportion of the buffer above threshold required in the actual system requires further study. Our simulations show that the required buffer size is related to the ratio of the buffer size below threshold to that above threshold, and is also related to the packet size distribution. In fact, an analysis of the buffer size above threshold has now been undertaken in [8]. Further work can be done to show that the Packet Discard Strategy may also be used in conjunction with the other congestion management strategies suggested in the literature for Available Bit Rate (ABR) traffic [16]-[19] to reduce cell loss multiplication.

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# Appendix A: The number of arrivals during a service period

From (8), we have

$$\begin{aligned} V(z) &= E\left[z^{G_1+G_2+\ldots+G_N}\right] \\ &= \sum_{n=0}^{\infty} P\left[N=n\right] E\left[z^{G_1+G_2+\ldots+G_n}\right] \\ &= \sum_{n=0}^{\infty} P\left[N=n\right] E\left[z^{G}\right]^{n} \end{aligned} \tag{134}$$

Substituting (7) into (134), we have

$$\begin{aligned} V(z) &= \sum_{n=0}^{\infty} \left( \int_0^{\infty} \left( \frac{(\lambda x)^n}{n!} e^{-\lambda x} \right) b(x)\, dx \right) E\left[z^{G}\right]^{n} \\ &= \int_0^{\infty} e^{-\lambda x} \sum_{n=0}^{\infty} \frac{(\lambda x E\left[z^{G}\right])^{n}}{n!} b(x)\, dx \\ &= \int_0^{\infty} e^{-\lambda x\left[1-G(z)\right]} b(x)\, dx \\ &= B^{*}(\lambda - \lambda G(z)) \end{aligned} \tag{135}$$

With constant service period $x$, we obtain

$$V(z) = e^{-\lambda \bar{x}\left[1-G(z)\right]} \tag{136}$$

# Appendix B: Derivation of $d_o$ using L'Hospital's rule

Applying L'Hospital's rule to (29) yields

$$Q^d(z)\big|_{z=1} = d_0 \left.\frac{V'(z)\,[1 - G(z)] - G'(z)V(z)}{V'(z) - 1}\right|_{z=1} = 1 \qquad (137)$$

Recall that

$$G(z)\big|_{z=1} = 1 \qquad G'(z)\big|_{z=1} = g \qquad (138)$$

Further, we have

$$V(z) = B^*(\lambda - \lambda G(z)) \quad \Rightarrow \quad V(z)\big|_{z=1} = 1$$
$$V'(z)\big|_{z=1} = -\lambda G'(z) B^{*\prime}(\lambda - \lambda G(z))\big|_{z=1} \qquad (139)$$

Substituting (138) and (139) into (137) and equating it to 1 yields

$$d_0 = \frac{1 - \lambda \bar{x} g}{g} \qquad (140)$$

# Appendix C: Proof of $d_k = d_0 f_k,\ k \geq 0$

Obviously, the relationship given in (37) is satisfied for $k = 0$ with $f_0 = 1$ as $d_0 = d_0 \cdot 1$. Now, let us assume that the relationship given in (37) is satisfied for all natural numbers up to and including $k$. Then consider $d_{k+1}$ given in (36) as

$$\begin{aligned} d_{k+1} &= \frac{1}{\alpha_0}\left(d_k - \beta_k d_o - \sum_{i=1}^{k} \alpha_{k+1-i} d_i\right) \\ &= \frac{1}{\alpha_0}\left(d_0 f_k - \beta_k d_o f_0 - \sum_{i=1}^{k} \alpha_{k+1-i} d_0 f_i\right) \\ &= \frac{1}{\alpha_0}\left(f_k - \beta_k f_0 - \sum_{i=1}^{k} \alpha_{k+1-i} f_i\right) d_0 \\ &= d_0 f_{k+1} \end{aligned} \tag{141}$$

where

$$f_{k+1} = \frac{1}{\alpha_0}\left(f_k - \beta_k f_0 - \sum_{i=1}^{k} \alpha_{k+1-i} f_i\right) \tag{142}$$

which leads to the desired result by mathematical induction.

# Appendix D: Numerical stability of results

The recursion formula given in (38) may suffer from loss of significance in computer implementations of practical interest since the quantity inside the brackets is the difference of the positive quantity $f_k$, which becomes small for large $k$, and a positive sum of comparable magnitude. However, it is possible to transform the relation into a mathematically equivalent form which is highly stable and ideally suited for numerical computation [15](pp 16). To that end, we define

$$F(z) = \sum_{i=0}^{\infty} f_i z^i \qquad A(z) = \sum_{i=0}^{\infty} \alpha_i z^i \qquad B(z) = \sum_{i=0}^{\infty} \beta_i z^i \tag{143}$$

and introduce

$$\hat{\alpha}_k = 1 - \sum_{i=0}^{k} \alpha_i \qquad \hat{\beta}_k = 1 - \sum_{i=0}^{k} \beta_i \qquad k \geq 0 \tag{144}$$

Noting that for $0 \leq z < 1$

$$\sum_{i=0}^{\infty} \hat{\alpha}_i z^i = \hat{A}(z) = \frac{1 - A(z)}{1 - z} \qquad \sum_{i=0}^{\infty} \hat{\beta}_i z^i = \hat{B}(z) = \frac{1 - B(z)}{1 - z} \tag{145}$$

and multiplying the both sides of equation (38) by $z^{k+1}$ and summing over $k$ as

$$\sum_{k=0}^{\infty} f_{k+1} z^{k+1} = \frac{1}{\alpha_0} \sum_{k=0}^{\infty} \left[ f_k - \beta_k - \sum_{i=1}^{k} f_i \alpha_{k+1-i} \right] z^{k+1} \tag{146}$$

lead to

$$\begin{aligned}
\alpha_0 [F(z) - f_0] &= zF(z) - zB(z) - \sum_{k=1}^{\infty} \sum_{i=1}^{k} f_i \alpha_{k+1-i} z^{k+1} \\
&= zF(z) - zB(z) - \sum_{i=1}^{\infty} f_i z^i \sum_{k=i}^{\infty} \alpha_{k+1-i} z^{k+1-i} \\
&= zF(z) - zB(z) - [F(z) - f_0]\,[A(z) - \alpha_0]
\end{aligned} \tag{147}$$

which in turn, after an algebraic manipulation, yields

$$[z - A(z)]\,F(z) = zB(z) - A(z). \tag{148}$$

Noting that

$$[z - A(z)]\,F(z) = \frac{1 - A(z)}{1 - z} F(z) - F(z) \tag{149}$$

$$zB(z) - A(z) = \frac{1 - A(z)}{1 - z} - f_0 - z\frac{1 - B(z)}{1 - z} \tag{150}$$

an equivalent relationship to the one given in (148) may be obtained as

$$F(z) - f_0 = z\hat{B}(z) + [F(z) - f_0]\,\hat{A}(z). \tag{151}$$

Upon series expansion, we have

$$
\begin{aligned}
[F(z) - f_0]\hat{A}(z) &= \sum_{i=1}^{\infty} f_i z^i \sum_{j=0}^{\infty} \hat{\alpha}_j z^j \\
&= \sum_{k=1}^{\infty} \sum_{i=1}^{k} \hat{\alpha}_{k-i} f_i z^k \\
&= \hat{\alpha}_0 \sum_{k=1}^{\infty} f_k z^k + \sum_{k=2}^{\infty} \sum_{i=1}^{k-1} \hat{\alpha}_{k-i} f_i z^k
\end{aligned}
\tag{152}
$$

Noting that $\hat{\alpha}_0 = 1 - \alpha_0, f_0 = 1$ and substituting (152) into (151) yield

$$
\alpha_0 \sum_{k=1}^{\infty} f_k z^k = \sum_{k=1}^{\infty} \beta_{k-1} z^k + \sum_{k=2}^{\infty} \left( \sum_{i=1}^{k-1} \hat{\alpha}_{k-i} f_i \right) z^k \tag{153}
$$

from which we obtain the recursion formula

$$
\begin{aligned}
f_0 &= 1 \\
f_1 &= \frac{1}{\alpha_0} \beta_0 \\
&\vdots \\
f_k &= \frac{1}{\alpha_0} \left( \beta_{k-1} + \sum_{i=1}^{k-1} \hat{\alpha}_{k-i} f_i \right) \qquad k \geq 2
\end{aligned}
\tag{154}
$$

which is highly recommended for the accurate and efficient computation of $f_k$'s.

# Appendix E: Double transform of $p_k(x)$ - $P^*(z, s)$

For the sake of convenience, we define

$$\bar{p}_k(x) \equiv \frac{p_k(x)}{1 - B(x)} \qquad k \geq 1 \tag{155}$$

as the conditional joint density function on the condition that the service time $X > x$. Then the differential equations given by (47) simplify to

$$\frac{d}{dx}\bar{p}_k(x) + \lambda \bar{p}_k(x) = \lambda \sum_{i=1}^{k-1} g_{k-i} \bar{p}_i(x) \qquad k \geq 1. \tag{156}$$

Next, we introduce the generating function $\bar{P}(z, x)$ for the sequence $\{\bar{p}_k(x); k \geq 1\}$ by

$$\bar{P}(z, x) \equiv \sum_{k=1}^{\infty} \bar{p}_k(x) z^k. \tag{157}$$

Then we can write (156) as

$$\frac{\partial}{\partial x}\bar{P}(z, x) + \lambda [1 - G(z)] \bar{P}(z, x) = 0 \tag{158}$$

which obviously has the solution

$$\bar{P}(z, x) = \bar{P}(z, 0) e^{-\lambda [1 - G(z)] x} \qquad x \geq 0. \tag{159}$$

The function $\bar{P}(z, 0)$ can be determined from boundary conditions given in (48) and (49). First, we apply the definition of the conditional density function given in (155) to the boundary conditions to obtain

$$\lambda p_0 = \int_0^\infty \bar{p}_1(x) dB(x) \tag{160}$$

$$\bar{p}_k(0) = \lambda g_k p_0 + \int_0^\infty \bar{p}_{k+1}(x) dB(x) \qquad k \geq 1. \tag{161}$$

If we multiply (161) by $z^k$ and sum over $k$, we get

$$\begin{aligned} \bar{P}(z, 0) &= \lambda p_0 G(z) + \frac{1}{z}\int_0^\infty [\bar{P}(z, x) - \bar{p}_1(x)]\, dB(x) \\ &= -\lambda p_0 [1 - G(z)] + \frac{1}{z}\int_0^\infty \bar{P}(z, x) dB(x) \end{aligned} \tag{162}$$

where we also make use of (160) and note that

$$\bar{P}(z, 0) \equiv \sum_{k=1}^{\infty} \bar{p}_k(0) z^k. \tag{163}$$

Substituting (159) into (162), we have

$$\bar{P}(z, 0) = -\lambda p_0 [1 - G(z)] + \frac{1}{z}\bar{P}(z, 0)\int_0^\infty e^{-\lambda[1 - G(z)]} dB(x). \tag{164}$$

Solving for $\bar{P}(z, 0)$, we get

$$\bar{P}(z, 0) = \frac{z [1 - G(z)] \lambda p_0}{B^*(\lambda - \lambda G(z)) - z}. \tag{165}$$

Finally, substituting (165) into (159) yields

$$\bar{P}(z,x) = \frac{z\,[1-G(z)]\,\lambda p_0}{B^*(\lambda-\lambda G(z))-z} e^{-\lambda[1-G(z)]x} \tag{166}$$

which requires $p_0$ to be determined. Applying the conditional density function given in (155) to the normalization condition, we have

$$p_0 + \int_0^\infty \bar{P}(z,x)\big|_{z=1}[1-B(x)]\,dx = 1. \tag{167}$$

$\bar{P}(z,x)\big|_{z=1}$ may be calculated from (166) using the techniques presented in Appendix B. Consequently, we have

$$p_0 = 1-\lambda \bar{x} g = 1-\rho. \tag{168}$$

Finally, we define the double transform $P^*(z,s)$ for $p_k(x)$ as

$$\begin{aligned} P^*(z,s) &\equiv \sum_{k=1}^{\infty} z^k \int_0^\infty e^{-sx} p_k(x)\,dx \\ &= \int_0^\infty e^{-sx}\bar{P}(z,x)\,[1-B(x)]\,dx \end{aligned} \tag{169}$$

where we make use of definitions given in (155) and (157). Substituting (166) into (169) and using integration by parts, we obtain

$$P^*(z,s) = \frac{\lambda(1-\rho)\,z\,[1-G(z)]}{B^*(\lambda-\lambda G(z))-z}\,\frac{1-B^*(s+\lambda-\lambda G(z))}{s+\lambda-\lambda G(z)}. \tag{170}$$

# Appendix F: Derivation of $p_0^T$

To derive $p_0^T$, we first solve the system of differential equations in Section 4.2.1 to obtain $p_k^T(0)$. Then, by making use of the relationship between $p_k^T(0)$ and the probability that the system is busy, we solve for $p_0^T$. Firstly, let us integrate (75) and (78) in Section 4.2.1 from 0 to $\infty$ to get

$$p_1^T(0) = \lambda(p_0^T + p_1^T) \tag{171}$$

$$p_k^T(0) - p_{k-1}^T(0) = \lambda\left[p_k^T - p_0^T g_{k-1} - \sum_{i=1}^{k-1} p_i^T g_{k-i}\right] \qquad 2 \le k \le T-1 \tag{172}$$

$$p_k^T(0) - p_{k-1}^T(0) = \lambda\left[-p_0^T g_{k-1} - \sum_{i=1}^{T-1} p_i^T g_{k-i}\right] \qquad T \le k \le \infty \tag{173}$$

where we make use of the boundary conditions in (80) and we also assume $p_k^T(\infty) \equiv 0$.

By observing that ;

$$p_l^T(0) = p_1^T(0) + \sum_{k=2}^{l} [p_k^T(0) - p_{k-1}^T(0)] \qquad 2 \le l \le T-2$$

we can combine (171) and (172) to evaluate $p_l^T(0)$

$$p_l^T(0) = \lambda\left[\sum_{k=0}^{l} p_k^T - p_0^T \sum_{i=1}^{l-1} g_i - \sum_{i=1}^{l-1} p_i^T \sum_{k=1}^{l-i} g_k\right] \qquad 1 \le l \le T-1 \tag{174}$$

Similarly observing that

$$p_l^T(0) = p_{T-1}^T(0) + \sum_{k=T}^{l} [p_k^T(0) - p_{k-1}^T(0)] \qquad l \geq T$$

we can also rewrite (173) as

$$p_l^T(0) = \lambda\left[\sum_{k=0}^{T-1} p_k^T - p_0^T \sum_{i=1}^{l-1} g_i - \sum_{i=1}^{T-1} p_i^T \sum_{k=1}^{l-i} g_k\right] \qquad l \geq T \tag{175}$$

Next, we define the probability generating function of $p_k^T(0)$ as

$$p_k^T(z, 0) \equiv \sum_{k=1}^{\infty} p_k^T(0) z^k \tag{176}$$

To evaluate $p_k^T(z, 0)$, we make use of (173) to (175) to get

$$\begin{aligned}\frac{p_k^T(z, 0)}{\lambda} = {} & (p_0^T + p_1^T) z \\ & + \sum_{l=2}^{T-1} \sum_{k=0}^{l} p_k^T z^l - p_0^T \sum_{l=2}^{T-1} \sum_{i=1}^{l-1} g_i z^l - \sum_{l=2}^{T-1} \sum_{i=1}^{l-1} p_i^T \sum_{k=1}^{l-i} g_k z^l \\ & + \sum_{l=T}^{\infty} \sum_{k=0}^{T-1} p_k^T z^l - p_0^T \sum_{l=T}^{\infty} \sum_{i=1}^{l-1} g_i z^l - \sum_{l=T}^{\infty} \sum_{i=1}^{T-1} p_i^T \sum_{k=1}^{l-i} g_k z^l \, .\end{aligned} \tag{177}$$

Now, we consider the terms on the RHS of the above equation in three pairs. Firstly, we have

$$\sum_{l=2}^{T-1} \sum_{k=0}^{l} p_k^T z^l + \sum_{l=T}^{\infty} \sum_{k=0}^{T-1} p_k^T z^l = (p_0^T + p_1^T) \sum_{l=2}^{\infty} z^l + \sum_{k=2}^{T-1} p_k^T \sum_{l=k}^{\infty} z^l$$

$$= \frac{z^2 (p_0^T + p_1^T)}{1-z} + \frac{1}{1-z} \sum_{k=2}^{T-1} p_k^T z^k$$

$$= -(p_0^T + p_1^T) z - p_0^T + \frac{1}{1-z} \sum_{k=0}^{T-1} p_k^T z^k .$$

Next, we have

$$p_0^T \sum_{l=2}^{T-1} \sum_{i=1}^{l-1} g_i z^l + p_0^T \sum_{l=T}^{\infty} \sum_{i=1}^{l-1} g_i z^l = \frac{\lambda p_0^T z}{1-z} G(z).$$

Finally, we have

$$\sum_{l=2}^{T-1}\sum_{i=1}^{l-1}p_i^T\sum_{k=1}^{l-i}g_kz^l+\sum_{l=T}^{\infty}\sum_{i=1}^{T-1}p_i^T\sum_{k=1}^{l-i}g_kz^l=\sum_{i=1}^{T-1}p_i^T\sum_{l=i+1}^{\infty}\sum_{k=1}^{l-i}g_kz^l$$

$$=\sum_{i=1}^{T-1}p_i^T\sum_{k=1}^{\infty}g_k\sum_{l=k+i}^{\infty}z^l$$

$$=\frac{G(z)}{1-z}\left[\sum_{i=0}^{T-1}p_i^Tz^i-p_0^T\right].$$

Accordingly, we can rewrite (177) as

$$\frac{p_k^T(z,0)}{\lambda}=-p_0^T+\frac{1}{1-z}\sum_{k=0}^{T-1}p_k^Tz^k-\frac{p_0^Tz}{1-z}G(z)+\frac{G(z)p_0^T}{1-z}-\frac{G(z)}{1-z}\sum_{k=0}^{T-1}p_k^Tz^k$$

$$\Rightarrow p_k^T(z,0)=[1-G(z)]\left[\frac{\lambda}{1-z}\sum_{k=0}^{T-1}p_k^Tz^k-\lambda p_0^T\right] \tag{178}$$

Note that

$$\sum_{k=1}^{\infty}\bar{x}p_k^T(0)=Pr\,[\text{the server is busy}]$$

The server can either be busy or not busy. Accordingly we have

$$p_0^T + \sum_{k=1}^{\infty} \bar{x} p_k^T(0) = 1, \tag{179}$$

which is also derived in [12].

Substituting (178) into (179), we have

$$p_0^T + \bar{x}\,[1 - G(z)]\left[\frac{\lambda}{1-z}\sum_{k=0}^{T-1} p_k^T z^k - \lambda p_0^T\right]\Bigg|_{z\to 1} = 1,$$

$$\Rightarrow p_0^T + \lambda\bar{x}\left[\frac{[1 - G(z)]\left(\sum_{k=0}^{T-1} p_k^T z^k - (1-z)\,p_0^T\right)}{1-z}\right]\Bigg|_{z\to 1} = 1. \tag{180}$$

Applying L'Hospital's rule to (180), we obtain

$$p_0^T + \lambda\bar{x}\left[\frac{[1 - G(z)]\left(\sum_{k=0}^{T-1} p_k^T z^k - (1-z)\,p_0^T\right)' - g\left(\sum_{k=0}^{T-1} p_k^T z^k - (1-z)\,p_0^T\right)}{-1}\right]\Bigg|_{z\to 1} = 1$$

$$\Rightarrow p_0^T + \lambda\bar{x} g \sum_{k=0}^{T-1} p_k^T = 1. \tag{181}$$

Substituting $p_k^T = p_0^T h_k$ as in (82) of Section 4.2.2 onto (181), we get

$$p_0^T + \lambda \bar{x} g p_0^T \sum_{k=0}^{T-1} h_k = 1$$

$$\Rightarrow p_0^T = \left[ 1 + \lambda \bar{x} g \sum_{k=0}^{T-1} h_k \right]^{-1} . \tag{182}$$

# Appendix G: Derivation of $p_0^B$

To derive $p_0^B$, we follow the same procedure as in the derivation of $p_0^T$. That is, we first solve the system of differential equations in Section 4.2.1 to obtain $p_k^B(0)$. Then, by making use of the relationship between $p_k^B(0)$ and the probability that the system is busy, we solve for $p_0^B$. Firstly, let us integrate (102) in Section 5.2.1 from 0 to ∞ to get

$$p_1^B(0) = \lambda (p_0^B + p_1^B) . \tag{183}$$

$$p_k^B(0) - p_{k-1}^B(0) = \lambda \left[ p_k^B - p_0^B g_{k-1} - \sum_{i=1}^{k-1} p_i^B g_{k-i} \right] \qquad 2 \le k < B, \tag{184}$$

where we make use of the boundary conditions in (106) and we also assume $p_k^B(\infty) \equiv 0$.

We can combine (183) and (184) into

$$p_k^B(0) = \lambda \left[ p_0^B g_k + \sum_{i=0}^{k} p_i^B - \sum_{i=0}^{k-1} p_i^B \sum_{j=1}^{k-i} g_j \right] \qquad 1 \le k < B. \tag{185}$$

From (107), we also have

$$p_B^B(0) = \sum_{i=B}^{\infty} \lambda g_i p_0^B . \tag{186}$$

Note that

$$\sum_{k=1}^{B} \bar{x} p_k^B(0) = Pr\,[\text{the server is busy}]\,,$$

$$p_0^B + \sum_{k=1}^{B} \bar{x} p_k^B(0) = 1, \tag{187}$$

which is also derived in in [12].

We can now substitute (185) and (186) into (187) and get

$$p_0^B + \lambda\bar{x} \sum_{k=1}^{B-1} \left[ p_0^B g_k + \sum_{i=0}^{k} p_i^B - \sum_{i=0}^{k-1} p_i^B \sum_{j=1}^{k-i} g_j \right] + \lambda\bar{x} \sum_{i=B}^{\infty} g_i p_0^B = 1 \tag{188}$$

$$\Rightarrow p_0^B + \lambda\bar{x} \left[ p_0^B + (B-1) p_0^B + \sum_{i=1}^{B-1} (B-i) p_i^B - \sum_{i=0}^{B-2} p_i^B \sum_{j=1}^{B-i-1} (B-i-j) g_j \right] = 1$$

$$\Rightarrow p_0^B + \lambda\bar{x} \left[ p_0^B \left( B - \sum_{j=1}^{B-1} (B-j) g_j \right) + \sum_{i=1}^{B-1} p_i^B \left( B - i + \sum_{j=1}^{B-i-1} j g_j - (B-i) \sum_{j=1}^{B-i-1} g_j \right) \right] = 1$$

$$\Rightarrow p_0^B + \lambda\bar{x} \left[ p_0^B \left( \sum_{i=1}^{B-1} i g_i + B \sum_{j=B}^{\infty} g_j \right) + \sum_{k=1}^{B-1} p_k^B \left( \sum_{j=1}^{B-k-1} j g_j + (B-k) \sum_{j=B-k}^{\infty} g_j \right) \right] = 1$$

$$\Rightarrow p_0^B + \lambda\bar{x} \sum_{k=0}^{B-1} p_k^B \left( \sum_{i=1}^{B-k-1} i g_i + (B-k) \sum_{i=B-k}^{\infty} g_i \right) = 1$$

$$\Rightarrow p_0^B = \left[1 + \lambda\bar{x}\sum_{k=0}^{B-1} h_k\left(\sum_{i=1}^{B-k-1} ig_i + (B-k)\sum_{i=B-k}^{\infty} g_i\right)\right]^{-1}$$

where we make use of $p_k^B = p_0^B h_k$ from (109). Then, by re-arranging, we have

$$p_0^B = \left[1 + \lambda\bar{x}\sum_{k=0}^{B-1} h_k\left(\sum_{i=1}^{B-k-1} ig_i + \sum_{i=B-k}^{\infty} ig_i - \sum_{i=B-k}^{\infty} ig_i + (B-k)\sum_{i=B-k}^{\infty} g_i\right)\right]^{-1},$$

$$\Rightarrow p_0^B = \left[1 + \lambda\bar{x}\sum_{k=0}^{B-1} h_k\left(\sum_{i=1}^{\infty} ig_i - \sum_{i=B-k}^{\infty} g_i[i-(B-k)]\right)\right]^{-1},$$

$$\Rightarrow p_0^B = \left[1 + \lambda\bar{x}\sum_{k=0}^{B-1} h_k\left(g - \left((B-k)g_{B-k} - (B-k)g_{B-k} + \sum_{i=B+1-k}^{\infty} g_i[i-(B-k)]\right)\right)\right]^{-1},$$

$$\Rightarrow p_0^B = \left[1 + \lambda\bar{x}g\sum_{k=0}^{B-1} h_k\left(1 - \sum_{i=B+1-k}^{\infty} g_i\frac{i-(B-k)}{g}\right)\right]^{-1}.$$